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Infinitesimal Calculus of Variations

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It is desirable that physical laws should be formulated infinitesimally, while it is well known that the calculus of variations, which has long been concerned with local or global horizons, gives a unifying viewpoint of various arenas of modern physics. The principal objective of this paper is to infinitesimalize the calculus of variations by making use of the vanguard of modern differential geometry, namely, synthetic differential geometry, in which nilpotent infinitesimals of various orders are abundantly and coherently available. Our treatment is completely coordinate-free, the decomposition of a state into its position and velocity components being replaced by the vertical–horizontal decomposition associated with an appropriate connection. Within our newly established infinitesimal calculus of variations, generalized conservation laws of momentum and energy are demonstrated.

INTRODUCTION

The variational viewpoint gives various fields of modern physics a unifying tone. By way of example, Snell's law in geometric optics, Lagrange's equation in analytical mechanics, and Schrödinger's equation in quantum mechanics can be put under the same umbrella of a unifying variational principle. Indeed, as is well known, it was the variational analogy between the mechanics of a point mass in a force field and geometric optics in an inhomogeneous medium that led Schrödinger to his splendid discovery of the wave theory of matter.

However, it is desirable that physical laws should be formulated infinitesimally, which would be tantamount to saying that physical laws should be expressed in terms of differential equations, were we to adhere to standard mathematics without infinitesimals at all. It is indeed mysterious that nature should obey variational principles, but it would be less mysterious if they were formulated infinitesimally.

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Synthetic differential geometry is an avant-garde branch of differential geometry, in which nilpotent infinitesimals, once ostracized from orthodox differential geometry, are abundantly and coherently available. The idea of nilpotent infinitesimals is by no means quixotic, but is indispensable in the well-established tradition of algebraic geometry à la Grothendieck. The principal objective of this paper is to infinitesimalize the calculus of variations within the burgeoning realm of synthetic differential geometry.

Microlinear spaces occupy such a central position in synthetic differential geometry as smooth manifolds have long enjoyed in standard differential geometry. Bunge and Heggie's (1984) synthetic approach to the calculus of variations is too narrow to cover microlinear spaces without a stitch of coordinates, inheriting too much of the legacy of coordinate manipulations from the standard calculus of variations. The missing link in a synthetic calculus of variations is provided by the vertical–horizontal decomposition associated with an appropriate pointwise connection. This point will be enlarged upon in Section 2, where an abstract version of Lagrange's equation is established. In Section 3 we will establish conservation laws of momentum and energy within our infinitesimal calculus of variations. Section 1 is devoted to miscellaneous preliminaries.

1. PRELIMINARIES

We assume that the reader is familiar with Lavendhomme's (1996) celebrated textbook on synthetic differential geometry up to Chapter 5. Since our discussions will be carried out within this synthetic framework, he or she should make it a rule to think not classically, but intuitionistically. However, except for abandonment of the principle of excluded middle and Zorn's lemma, he or she can presume that we are working within the standard universe of sets. The set of real numbers (including nilpotent infinitesimals in abundance) is denoted by $\mathbb{R}$ and is required to abide by the so-called general Kock axiom (Lavendhomme, 1996, §2.1.3), which surely subsumes the Kock–Lawvere axiom (Lavendhomme, 1996, §§1.1.1) as the fulcrum of synthetic differential calculus. The set of natural numbers is denoted by $\mathbb{N}$. We denote the sets $\{d \in \mathbb{R} | d^2 = 0\}$ and $\{e \in \mathbb{R} | e^n = 0 \text{ for some } n \in \mathbb{N}\}$ by D and D_∞, respectively. The Kock–Lawvere axiom guarantees that for any function $f\colon D \to \mathbb{R}$ there exists a unique $\delta f \in \mathbb{R}$ such that

$$(1.1)\quad f(d) - f(0) = d\delta f$$

for any $d \in D$. Given a function $g\colon D_\infty \to \mathbb{R}$, we denote by g' the function assigning $\delta g(e + \cdot)$ to each $e \in D_\infty$, where $g(e + \cdot)$ denotes the function assigning $g(e + d)$ to each $d \in D$.

Our axiom of integration goes as follows:

(1.2) For any function f: $D_\infty \to \mathbb{R}$, there exists a unique function g: $D_\infty \to \mathbb{R}$ with $g' = f$ and $g(0) = 0$.

Given $e_1, e_2 \in D_\infty$, we denote $g(e_2) - g(e_1)$ by $\int_{e_1}^{e_2} f(e)\mathbf{d}e$ or $\int_{e_1}^{e_2} f$. The following simple but striking proposition on integration is well known in synthetic infinitesimal calculus and will be useful in Section 3.

Proposition 1.1. For any function f: $D_\infty \to \mathbb{R}$, any $e \in D_\infty$, and any $d \in D$, we have

(1.3) $\int_e^{e+d} f = df(e)$

Proof. See Proposition 11 of Lavendhomme (1996, §1.3). ■

Now we turn to connections on vector bundles. A microlinear space M shall be chosen once and for all in this section. A *vector bundle over M* is a mapping ξ: $E \to M$ such that E is a microlinear space and $E_x = \xi^{-1}(x)$ is a Euclidean $\mathbb{R}$-module for any $x \in M$. By way of example, the tangent bundle τ_M: $M^D \to M$ [i.e., $\tau_M(t) = t(0)$ for any $t \in M^D$] and the trivial bundle π_1: $M \times \mathcal{A} \to M$ for a Euclidean $\mathbb{R}$-module $\mathcal{A}$ [i.e., $\pi_1(x, a) = x$ for any $(x, a) \in M \times \mathcal{A}$] are vector bundles over M. Given vector bundles ξ: $E \to M$ and η: $F \to M$ over M, we denote by $\mathcal{L}(\xi, \eta)$ the totality of linear mappings from E_x to F_x for $x \in M$, and the mapping $\pi_{\mathcal{L}(\xi,\eta)}$: $\mathcal{L}(\xi, \eta) \to M$ assigning x to each linear mapping from E_x to F_x is a vector bundle over M. In particular, if η is a trivial bundle π_1: $M \times \mathcal{A} \to M$ for a Euclidean $\mathbb{R}$-module $\mathcal{A}$, then $\mathcal{L}(\xi, \eta)$ and $\pi_{\mathcal{L}(\xi,\eta)}$ are also denoted by $\mathcal{L}(\xi, \mathcal{A})$ and $\pi_{\mathcal{L}(\xi,\mathcal{A})}$, respectively.

Let ξ: $E \to M$ be a vector bundle. We denote by K_ξ the mapping which assigns, to each $t \in E^D$, $(\xi \circ t, t(0)) \in M^D \times_M E$. Both E^D and $M^D \times_M E$ can be regarded naturally as vector bundles over E and over M^D, and K_ξ is linear with respect to both vector bundle structures (Moerdijk and Reyes (1991, Chapter V. Proposition 3.4.8)). A (*linear*) *connection* on ξ is a mapping ∇: $M^D \times_M E \to E^D$ pursuant to the following conditions:

(1.4) It is a section of K_ξ, i.e., $K_\xi \circ \nabla$ is the identity transformation of $M^D \times_M E$.

(1.5) It is homogeneous with respect to both vector bundle structures $\odot$ over E and $\cdot$ over M^D.

(1.6) For any $x \in M$ and any $(t,d) \in M^D \times D$, the mapping $u \in E_x \mapsto \nabla(t, u)(d) \in E_{t(d)}$, denoted by $p^{\nabla}_{(t,d)}$ or $p_{(t,d)}$, is bijective. Its inverse is denoted by $q^{\nabla}_{(t,d)} = q_{(t,d)}$: $E_{t(d)} \to E_x$. We call $p_{(t,d)}$ the *parallel transport* from $t(0)$ to $t(d)$ along t, while $q_{(t,d)}$ is called the *parallel transport* from $t(d)$ to $t(0)$ along t.

If the vector bundle ξ: $E \to M$ is a trivial bundle $M \times \mathcal{A} \to M$ and

$\nabla(t, (t(0),a))(d) = (t(d), a)$ for any $t \in M^D$, any $a \in \mathcal{A}$, and any $d \in D$, then the connection ∇ is called *trivial.* A connection on the tangent bundle τ_M: $M^D \to M$ is called a *connection on M*. From now on in this section we will assume that M is bestowed with a preassigned connection ∇_M.

Synthetic differential calculus for $\mathbb{R}$-valued functions (Lavendhomme, 1996, §§1.1, 1.2) can be generalized easily, to a certain extent, to synthetic covariant differential calculus for E-valued functions, where ξ: $E \to M$ is a vector bundle over M and it is endowed with a connection ∇. By way of example, we have a variant of (1.4) claiming that for any function t: $D \to E$ there exists a unique $\delta t \in E_{\xi(\bar{t}(0))}$ such that

$$(1.7) \quad q^{\nabla}_{(t,d)}\,(\bar{t}(d)) - \bar{t}(0) = d\delta\bar{t}$$

for any $d \in D$, where $t = \xi \circ \bar{t}$. In particular, if the vector bundle happens to be the tangent bundle τ_M: $M^D \to M$ with $\nabla = \nabla_M$, then a microsquare γ on M can be reckoned as a function $\gamma(?, \cdot)$ assigning $\gamma(d, \cdot) \in M^D$ to each $d \in D$, and $\delta\gamma(?, \cdot)$ coincides with $C(\gamma)$ by Proposition 7 of Lavendhomme (1996, §5.2).

Now we conclude this section by discussing induced connections. Let ξ: $E \to M$ and η: $F \to M$ be vector bundles over the same base space M with connections ∇ and ∇' bestowed upon them. We now define an induced connection $\hat{\nabla}$ on $\pi_{\mathcal{L}(\xi,\eta)}$ as follows:

(1.8) $\hat{\nabla}(t, \hat{v})(d)(v) = p^{\nabla'}_{(t,d)}(\hat{v}(q^{\nabla}_{(t,d)}(v)))$ for any $t \in M^D$, any $d \in D$, any $\hat{v} \in \mathcal{L}(\xi, \eta)_{t(0)}$, and any $v \in E_{t(d)}$.

Proposition 1.2. For any $\sigma \in \mathcal{L}(\xi, \eta)^D$ and any $\gamma \in E^D$ with $(\pi_{\mathcal{L}(\xi,\eta)})^D(\sigma) = \xi^D(\gamma)$, we have

$$(1.9) \quad \delta(\sigma(\gamma)) = (\delta\sigma)(\gamma(0)) + \sigma(0)(\delta\gamma)$$

where $\sigma(\gamma)$ denotes the mapping $d \in D \mapsto \sigma(d)(\gamma(d))$.

Proof. Let $t = (\pi_{\mathcal{L}(\xi,\eta)})^D(\sigma) = \xi^D(\gamma)$. For any $d \in D$, we have

$$\begin{aligned}(1.10) \quad & q^{\nabla'}_{(t,d)}\,(\sigma(d)\,(\gamma(d))) \\ &= q^{\hat{\nabla}}_{(t,d)}\,(\sigma(d))(q^{\nabla}_{(t,d)}\,(\gamma(d))) \\ &= (\sigma(0) + d\delta\sigma)(\gamma(0) + d\delta\gamma) \\ &= \sigma(0)(\gamma(0)) + d\{(\delta\sigma)(\gamma(0)) + \sigma(0)\,(\delta\gamma)\}\end{aligned}$$

Therefore the desired proposition obtains. ■

2. LAGRANGE'S EQUATIONS

A microlinear space M ornamented with a symmetric connection ∇ shall be fixed once and for all in this section. Let us suppose also that a function

L on M^D to $\mathbb{R}$, called a *Lagrangian on M*, is given. We say that the Lagrangian L and the connection ∇ are *infinitesimally compatible* if they satisfy the following condition:

$$\text{(2.1)}\quad L(\gamma(d,\cdot)) - L(p_{(\gamma(\cdot,0),d)}(\gamma(0,\cdot))) = L(q_{(\gamma(\cdot,0),d)}(\gamma(d,\cdot))) - L(\gamma(0,\cdot))$$

for any microsquare γ on M, where $p_{(\gamma(\cdot,0),d)}$ and $q_{(\gamma(\cdot,0),d)}$ denote the mutually inverse parallel transports along the infinitesimal path $\gamma(\cdot,0)$ between $\gamma(0,0)$ and $\gamma(d,0)$ (Lavendhomme, 1996, §5.2.3).

Let us give two important examples of the above situation in which (2.1) obtains.

Proposition 2.1. If M is a Euclidean $\mathbb{R}$-module E so that M^D can be identified with $E \times E$ [$(a,b) \in E \times E$ naturally gives rise to a mapping $d \in D \mapsto (a + db) \in E$], and if the connection ∇ is trivial in the sense that

$$\text{(2.2)}\quad \nabla((a,b),(a,c))(d_1, d_2) = a + d_1 b + d_2 c$$

for any $a,b,c \in E$ and any $d_1,d_2 \in D$, then (2.1) obtains.

Proof. Since E is a Euclidean $\mathbb{R}$-module, any microsquare γ on E is of the following form for unique $a_1,a_2,b_1,b_2 \in E$:

$$\text{(2.3)}\quad \gamma(d_1, d_2) = a_1 + d_1 a_2 + d_2(b_1 + d_1 b_2)$$

for any $(d_1,d_2) \in D^2$. Therefore we have

$$\begin{aligned}\text{(2.4)}\quad & L(\gamma(d,\cdot)) - L(p_{(\gamma(\cdot,0),d)}(\gamma(0,\cdot)))\\ &= L(a_1 + da_2, b_1 + db_2) - L(a_1 + da_2, b_1)\\ &= \{L(a_1, b_1) + d\partial^1_{a_2} L(a_1,b_1) + d\partial^2_{b_2} L(a_1,b_1)\}\\ &\quad -\{L(a_1, b_1) + d\partial^1_{a_2} L(a_1, b_1)\}\\ &= d\partial^2_{b_2} L(a_1, b_1)\end{aligned}$$

where $\partial^1_{a_2} L(a_1,b_1)$ denotes the derivative of the mapping $d \in D \mapsto L(a_1 + da_2,b_1)$ at 0, while $\partial^2_{b_2} L(a_1,b_1)$ denotes the derivative of the mapping $d \in D \mapsto L(a_1,b_1 + db_2)$ at 0. We have also

$$\begin{aligned}\text{(2.5)}\quad & L(q_{(\gamma(\cdot,0),d)}(\gamma(d,\cdot))) - L(\gamma(0,\cdot))\\ &= L(a_1,b_1 + db_2) - L(a_1, b_1)\\ &= \{L(a_1,b_1) + d\partial^2_{b_2} L(a_1,b_1)\} - L(a_1,b_1)\\ &= d\partial^2_{b_2} L(a_1, b_1)\end{aligned}$$

Therefore (2.1) obtains. ■

Proposition 2.2. If the Lagrangian L is covariantly invariant with respect to the connection ∇ (e.g., L is represented by a covariantly invariant metric g on M), then (2.1) obtains.

From now on we will assume that (2.1) obtains. We now define two types of partial differentiation of L. Given $s,t \in M^D$ with $s(0) = t(0)$, there exists unique $\mathbf{D}_1 L(t)(s) \in \mathbb{R}$ such that

$$L(p_{(s,d)}(t)) - L(t) = d\mathbf{D}_1 L(t)(s) \tag{2.6}$$

for any $d \in D$.

Proposition 2.3. Given $t \in M^D$ with $x = t(0)$, the mapping $s \in (M^D)_x \mapsto \mathbf{D}_1 L(t)(s)$ is homogeneous.

Proof. For any $\alpha \in \mathbb{R}$ we have

$$\begin{aligned} (2.7)\quad & d\mathbf{D}_1 L(t)(\alpha s) \\ &= L(p_{(\alpha s,d)}(t)) - L(t) \\ &= L(p_{(s,\alpha d)}(t)) - L(t) \\ &= \alpha d\mathbf{D}_1 L(t)(s) \end{aligned}$$

Therefore the desired statement follows.

We now turn to the other type of partial differentiation of L. Given $s,t \in M^D$ with $s(0) = t(0)$, there exists unique $\mathbf{D}_2 L(t)(s) \in \mathbb{R}$ such that

$$L(t + ds) - L(t) = d\mathbf{D}_2 L(t)(s) \tag{2.8}$$

for any $d \in D$.

Proposition 2.4. Given $t \in M^D$ with $x = t(0)$, the mapping $s \in (M^D)_x \mapsto \mathbf{D}_2 L(t)(s)$ is homogeneous.

Proof. For any $\alpha \in \mathbb{R}$ we have

$$\begin{aligned} (2.9)\quad & d\mathbf{D}_2 L(t)(\alpha s) \\ &= L(t + d\alpha s) - L(t) \\ &= \alpha d\mathbf{D}_2 L(t)(s) \end{aligned}$$

Therefore the desired statement follows. ■

Proposition 2.5. For any $\gamma \in M^{D^2}$ and any $d \in D$, we have

$$\begin{aligned} (2.10)\quad & L(\gamma(d,\cdot)) - L(\gamma(0,\cdot)) \\ &= d\{\mathbf{D}_2 L(\gamma(0,\cdot))(\omega(\gamma)) + \mathbf{D}_1 L(\gamma(0,\cdot))(\gamma(\cdot,0))\} \end{aligned}$$

Proof. We have that

$$\begin{aligned} (2.11)\quad & L(\gamma(d,\cdot)) - L(\gamma(0,\cdot)) \\ &= L(\gamma(d,\cdot)) - L(p_{(\gamma(\cdot,0),d)}(\gamma(0,\cdot))) \\ &\quad + L(p_{(\gamma(\cdot,0),d)}(\gamma(0,\cdot))) - L(\gamma(0,\cdot)) \\ &= L(q_{(\gamma(\cdot,0),d)}(\gamma d,\cdot))) - L(\gamma(0,\cdot)) \\ &\quad + d\mathbf{D}_1 L(\gamma(0,\cdot))(\gamma(\cdot,0)) \quad [(2.1) \text{ and } (2.6)] \end{aligned}$$

$$= L(\gamma(0,\cdot) + dC(\gamma)) - L(\gamma(0,\cdot))$$
$$+ d\mathbf{D}_1 L(\gamma(0,\cdot))(\gamma(\cdot,0))$$
(Proposition 7 of Lavendhomme, 1996, §5.2)
$$= d\mathbf{D}_2 L(\gamma(0,\cdot))(C(\gamma)) + d\mathbf{D}_1 L(\gamma(0,\cdot))(\gamma(\cdot,0))$$
[(2.8)]

Therefore the desired result (2.10) obtains. ■

For any $\theta \in M^{D \times D_\infty}$ and any $e \in D_\infty$, we define $\partial\theta(e) \in M^{D^2}$ to be
(2.12) $(d_1,d_2) \in D^2 \mapsto \theta(d_1, e + d_2) \in M$

Then the next result follows from Proposition 7 of §5.2 and Corollary 5 of §5.3 of Lavendhomme (1996):

Proposition 2.6. We have
(2.13) $\delta(\partial\theta(e)(?,\cdot)) = \delta(\partial\theta(e)(\cdot,?))$

where $\partial\theta(e)(?,\cdot)$ denotes the function $d \in D \mapsto \partial\theta\ (e)(d,\cdot) \in M^D$, while $\partial\theta(e)(\cdot,?)$ denotes the function $d \in D \mapsto \partial\theta(e)(\cdot,d) \in M^D$.

Proposition 2.7. For any $\theta \in M^{D \times D_\infty}$ and any $e \in D_\infty$, we have

(2.14) $L\,(\partial\theta(e)\ (d(,\cdot)) - L\,(\partial\theta(e)(0,\cdot))$
$$= d\{((\mathbf{D}_2 L\,(\partial\theta(?)(0,\cdot)))(\partial\theta(?)(\cdot,0)))'(e)$$
$$- ((\mathbf{D}_2 L(\partial\theta(?)(0,\cdot)))'\ (e))(\partial\theta(e)(\cdot,0))$$
$$+ \mathbf{D}_1 L(\partial\theta(e)(\cdot,0))(\partial\theta(e)(\cdot,0))\}$$

where $((\mathbf{D}_2 L(\partial\theta(?)(0,\cdot)))(\partial\theta(?)\ (\cdot,0)))'(e)$ denotes the derivative of the mapping

$$e' \in D_\infty \mapsto (\mathbf{D}_2 L(\partial\theta\ (e')(0,)))(\partial\theta(e')(\cdot,0)) \in \mathbb{R} \text{ at } e$$

and $(\mathbf{D}_2 L(\partial\theta(?)(0,\cdot)))'(\mathrm{e})$ denotes the covariant derivative of the mapping $e' \in D_\infty \mapsto \mathbf{D}_2 L(\partial\theta(e')(0,\cdot)) \in \mathcal{L}\,(\tau_M, \mathbb{R})$ at e.

Proof. We have

(2.15) $L(\partial\theta(e)(d,\cdot)) - L\,(\partial\theta(e)(0,\cdot))$
$$= d\{\mathbf{D}_2 L(\partial\theta(e)(0,\cdot))\ (\delta(\partial\theta(e)(?,\cdot)))$$
$$+\mathbf{D}_1 L(\partial\theta(e)(0,\cdot))(\partial\theta\ (e)(\cdot,0))\} \quad \text{[Proposition 2.5]}$$
$$= d\{\mathbf{D}_2 L(\partial\theta(e)(0,\cdot))\ (\delta(\partial\theta(e)(\cdot,?)))$$
$$+\mathbf{D}_1 L(\partial\theta(e)(0,\cdot))(\partial\theta(e)(\cdot,0))\ \} \quad \text{[Proposition 2.6]}$$
$$= d\{\mathbf{D}_2 L(\partial\theta(e)(0,\cdot))\ \delta(\partial\theta(e + ?)(\cdot,0)))$$
$$+\mathbf{D}_1 L(\partial\theta(e)(0,\cdot))(\partial\theta(e)(\cdot,0))\}$$
[since $\partial\theta(e + ?)(\cdot,0) = \theta(\cdot, e + ?) = \partial\theta(e)(\cdot,?)$]
$$= d\{((\mathbf{D}_2 L(\partial\theta(?)(0,\cdot)))(\partial\theta(?)\ (\cdot,0)))'\ (e)$$
$$-((\mathbf{D}_2 L(\partial\theta(?)(0,\cdot)))'(e))\ (\partial\theta(e)(\cdot,0))$$

$$+\mathbf{D}_1 L(\partial\theta(e)(0,\cdot))(\partial\theta(e)(\cdot,0))\} \quad \text{[Proposition 1.2]}$$

Therefore the desired proposition obtains. ■

Now let us assume the following *fundamental axiom of infinitesimal calculus of variations*:

(2.16) Given $F: D_\infty \to \mathscr{L}(\tau_M, \mathbb{R})$, if we have

(2.16.1) $$\int_{e_1}^{e_2} F(e)(f(e))\mathbf{d}e = 0$$

for any $e_1, e_2 \in D_\infty$ and any $f: \mathrm{D}_\infty \to M^{\mathrm{D}}$ such that

(2.16.2) $$(\pi_{\mathscr{L}(\tau_M,\mathbb{R})})^{D_\infty}(F) = (\tau_M)^{D_\infty}(\mathrm{f})$$

and

(2.16.3) both $f(e_1)$ and $f(e_2)$ vanish

then F vanishes all over D_∞.

The following is an abstract version of Lagrange's equation.

Theorem 2.8. We assume (2.16). Given $\alpha: D_\infty \to M$, if we have

(2.17) $$\delta\int_{e_1}^{e_2} L(\partial\theta(e)(?,\cdot))\mathbf{d}e = 0$$

for any $e_1, e_2 \in D_\infty$ and any $\theta: D \times D_\infty \to M$ such that

(2.18) $$\alpha(\cdot) = \theta(0,\cdot)$$

and

(2.19) Both tangent vectors $\theta(\cdot,e_1)$ and $\theta(\cdot, e_2)$ to M vanish

then we have the following Lagrange equation:

(2.20) $$(\mathbf{D}_2 L(\alpha'(?)))'(e) - \mathbf{D}_1 L(\alpha'(e)) = 0$$

for any $e \in D_\infty$, where $(\mathbf{D}_2 L(\alpha'(?)))'(e)$ denotes the covariant derivative of the mapping $e' \in D_\infty \mapsto \mathbf{D}_2 L(\alpha'(e')) \in \mathscr{L}(\tau_M, \mathbb{R})$ at e.

Proof. For any $d \in D$ we have

(2.21) $$0$$

$$= d\delta\int_{e_1}^{e_2} L(\partial\theta(e)(?, \cdot))\mathbf{d}e \qquad [(2.17)]$$

$$= \int_{e_1}^{e_2} L(\partial\theta(e)(d, \cdot))\mathbf{d}e - \int_{e_1}^{e_2} L(\partial\theta(e)(0, \cdot))\mathbf{d}e$$

$$d\left\{\int_{e_1}^{e_2} ((\mathbf{D}_2 L(\partial\theta(?)(0, \cdot)))(\partial\theta(?)(\cdot, 0)))'(e)\mathbf{d}e \right.$$
$$- \int_{e_1}^{e_2} ((\mathbf{D}_2 L(\partial\theta(?)(0, \cdot)))'(e) - \mathbf{D}_1 L(\partial\theta(e)(0, \cdot)))$$
$$\left. \times (\partial\theta(e)(\cdot, 0))\mathbf{d}e \right\} \qquad \text{[Proposition 2.7]}$$

$$= d\left\{(\mathbf{D}_2 L(\partial\theta(e_2)(0, \cdot)))(\partial\theta(e_2)(\cdot, 0)) \right.$$
$$- (\mathbf{D}_2 L(\partial\theta(e_1)(0, \cdot)))(\partial\theta(e_1)(\cdot, 0))$$
$$- \int_{e_1}^{e_2} ((\mathbf{D}_2 L(\partial\theta(?)(0, \cdot)))'(e) - \mathbf{D}_1 L(\partial\theta(e)(0, \cdot)))$$
$$\left. \times (\partial\theta(e)(\cdot, 0))\mathbf{d}e \right\}$$

$$= -d \int_{e_1}^{e_2} ((\mathbf{D}_2 L(\partial\theta(?)(0, \cdot)))'(e) - \mathbf{D}_1 L(\partial\theta(e)(0, \cdot)))$$
$$\times (\partial\theta(e)(\cdot, 0))\mathbf{d}e \qquad \text{[(2.19)]}$$

$$= -d \int_{e_1}^{e_2} ((\mathbf{D}_2 L(\alpha'(?)))'(e) - \mathbf{D}_1 L(\alpha'(e)))(\theta(\cdot, e))\mathbf{d}e$$

Therefore (2.20) follows from the fundamental axiom of infinitesimal calculus of variations. ■

Any function $\alpha: D_\infty \to M$ obeying condition (2.20) is called a *Lagrangian flow on M*.

3. CONSERVATION LAWS

In this section we will elicit generalized conservation laws of momentum and energy from Proposition 2.7. We will continue to assume that M is a microlinear space endowed with a symmetric connection ∇, for which (2.1) obtains. First we deal with a generalized conservation law of momentum.

Theorem 3.1. Given $\theta: D \times D_\infty \to M$ and $e_1, e_2 \in D_\infty$, if $\theta(0,\cdot)$ is a Lagrangian flow, and if θ satisfies

$$(3.1) \quad \delta \int_{e_1}^{e_2} L(\partial\theta(e)(?, \cdot))\mathbf{d}e = 0$$

then we have

$$(3.2)\quad \mathbf{D}_2L(\partial\theta(e_1)(0,\ \cdot))(\partial\theta(e_1)(\cdot,\ 0)) = \mathbf{D}_2L(\partial\theta(e_2)(0,\ \cdot))(\partial\theta(e_2)(\cdot,\ 0))$$

Proof. For any $d \in D$ we have

$$\begin{aligned}(3.3)\quad 0 &= d\delta \int_{e_1}^{e_2} L(\partial\theta(e)(?,\ \cdot))\mathbf{d}e \qquad [(3.1)]\\
&= \int_{e_1}^{e_2} L(\partial\theta(e)(d,\ \cdot))\mathbf{d}e - \int_{e_1}^{e_2} L(\partial\theta(e)(0,\ \cdot))\mathbf{d}e\\
&= d\left\{\int_{e_1}^{e_2} ((\mathbf{D}_2L(\partial\theta(?)(0,\ \cdot)))(\partial\theta(?)(\cdot,\ 0)))'(e)\mathbf{d}e\right.\\
&\quad - \int_{e_1}^{e_2} ((\mathbf{D}_2L(\partial\theta(?)(0,\ \cdot)))'(e) - \mathbf{D}_1L(\partial\theta(e)(0,\ \cdot)))\\
&\quad \left.\times (\partial\theta(e)(\cdot,\ 0))\mathbf{d}e\right\} \qquad [\text{Proposition 2.7}]\\
&= d\{(\mathbf{D}_2L(\partial\theta(e_2)(0,\ \cdot)))(\partial\theta(e_2)(\cdot,\ 0))\\
&\quad - (\mathbf{D}_2L(\partial\theta(e_1)(0,\ \cdot)))(\partial\theta(e_1)(\cdot,\ 0))\}\\
&\quad [\text{since, } \theta(0,\ \cdot) \text{ is a Lagrangian flow}]\end{aligned}$$

Therefore (3.2) obtains. ■

We now turn to a generalized conservation law of energy.

Theorem 3.2. Given $\theta: D \times D_\infty \to M$ and $\varphi_1, \varphi_2: D \to D_\infty$, if $\theta(0, \cdot)$ is a Lagrangian flow, if $\theta(d, \varphi_1(d)) = \theta(0, \varphi_1(0))$ and $\theta(d, \varphi_2(d)) = \theta(0, \varphi_2(0))$ for any $d \in D$, and if θ satisfies

$$(3.4)\quad \delta \int_{\varphi_1(?)}^{\varphi_2(?)} L(\partial\theta(e)(?,\ \cdot))\mathbf{d}e = 0$$

then

$$\begin{aligned}(3.5)\quad & \delta\varphi_2\{\mathbf{D}_2L(\partial\theta(\varphi_2(0))(0,\ \cdot))(\partial\theta(\varphi_2(0))(0,\ \cdot))\\
&\quad - L(\partial\theta(\varphi_2(0))(0,\ \cdot))\}\\
&= \delta\varphi_1\{\mathbf{D}_2L(\partial\theta(\varphi_1(0))(0,\ \cdot))(\partial\theta(\varphi_1(0))(0,\ \cdot))\\
&\quad - L(\partial\theta(\varphi_1(0))(0,\ \cdot))\}\end{aligned}$$

where $\delta\int_{\varphi_1(?)}^{\varphi_2(?)} L(\partial\theta(e)(?,\ \cdot))\mathbf{d}e$ denotes the derivative of the mapping $d \in D \mapsto \int_{\varphi_1(d)}^{\varphi_2(d)} L(\partial\theta(e)(d,\ \cdot))\mathbf{d}e \in \mathbb{R}$ at 0.

Proof. We have that

$$(3.6)\quad \int_{\varphi_1(d)}^{\varphi_2(d)} L(\partial\theta(e)(d,\cdot))\mathbf{d}e - \int_{\varphi_1(0)}^{\varphi_2(0)} L(\partial\theta(e)(0,\cdot))\mathbf{d}e$$
$$= \int_{\varphi_2(0)}^{\varphi_2(d)} L(\partial\theta(e)(d,\cdot))\mathbf{d}e - \int_{\varphi_1(0)}^{\varphi_1(d)} L(\partial\theta(e)(d,\cdot))\mathbf{d}e$$
$$+ \int_{\varphi_1(0)}^{\varphi_2(0)} \{L(\partial\theta(e)(d,\cdot)) - L(\partial\theta(e)(0,\cdot))\}\mathbf{d}e$$

We have also that

$$(3.7)\quad \int_{\varphi_2(0)}^{\varphi_2(d)} L(\partial\theta(e)(d,\cdot))\mathbf{d}e$$
$$= \int_{\varphi_2(0)}^{\varphi_2(0)+d\delta\varphi_2} L(\partial\theta(e)(d,\cdot))\mathbf{d}e$$
$$= d\delta\varphi_2 L(\partial\theta(\varphi_2(0))(d,\cdot)) \quad \text{[Proposition 1.1]}$$
$$= d\delta\varphi_2\{L(\partial\theta(\varphi_2(0))(0,\cdot)) + d\delta(L(\partial\theta(\varphi_2(0))(?,\cdot)))\}$$
$$= d\delta\varphi_2 L(\partial\theta(\varphi_2(0))(0,\cdot))$$

By the same token we have

$$(3.8)\quad \int_{\varphi_1(0)}^{\varphi_1(d)} L(\partial\theta(e)(d,\cdot))\mathbf{d}e$$
$$= d\delta\varphi_1 L(\partial\theta(\varphi_1(0))(0,\cdot))$$

By the same token as in Theorem 3.1 we have

$$(3.9)\quad \int_{\varphi_1(0)}^{\varphi_2(0)} \{L(\partial\theta(e)(d,\cdot)) - L(\partial\theta(e)(0,\cdot))\}\mathbf{d}e$$
$$= \mathbf{D}_2L(\partial\theta(\varphi_2(0))(0,\cdot))(\partial\theta(\varphi_2(0))(\cdot,0))$$
$$- \mathbf{D}_2L(\partial\theta(\varphi_1(0))(0,\cdot))(\partial\theta(\varphi_1(0))(\cdot,0))$$

Since $\theta(d,\varphi_1(d)) = \theta(0,\varphi_1(0))$ for any $d \in D$ by assumption, we have

$$(3.10)\quad \partial\theta(\varphi_1(0))(\cdot,0) + \partial\theta(\varphi_1(0))(0,\cdot)\delta\varphi_1 = 0$$

By the same token we have

$$(3.11)\quad \partial\theta(\varphi_2(0))(\cdot,0) + \partial\theta(\varphi_2(0))(0,\cdot)\delta\varphi_2 = 0$$

It follows from (3.6)–(3.11) that

$$(3.12)\quad \int_{\varphi_1(d)}^{\varphi_2(d)} L(\partial\theta(e)(d,\cdot))\mathbf{d}e - \int_{\varphi_1(0)}^{\varphi_2(0)} L(\partial\theta(e)(0,\cdot))\mathbf{d}e$$
$$= d\delta\varphi_2\{\mathbf{D}_2L(\partial\theta(\varphi_2(0))(0,\cdot))(\partial\theta(\varphi_2(0))(0,\cdot))$$
$$- L(\partial\theta(\varphi_2(0))(0,\cdot))\}$$

$$- d\delta\varphi_1\{\mathbf{D}_2L(\partial\theta(\varphi_1(0))(0, \cdot))(\partial\theta(\varphi_1(0))(0, \cdot)) - L(\partial\theta(\varphi_1(0))(0, \cdot))\}$$

Our desired result (3.5) now follows from (3.4) and (3.12).

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